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Eureka! Info-Gap is Worst-Case Analysis (Maximin) in Disguise!

Moshe Sniedovich
Department of Mathematics and Statistics
The University of Melbourne
Melbourne, Vic 3010, Australia

moshe@ms.unimelb.edu.au
www.ms.unimelb.edu.au/∼moshe

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Abstract

Info-Gap is supposedly a new theory for decision making under severe uncertainty. Its claim to fame is that it is non-probabilistic in nature and thus offers an alternative to all current theories for decision making under uncertainty. In this short article I show that *Info-Gap* is neither new nor radically different from current decision theories. Specifically, I formally prove that *Info-Gap*'s decision theoretic model is a simple instance of *Wald's Maximin Principle*, the most celebrated Principle in decision making under strict uncretainty.

Keywords: Decision making, severe uncertainty, maximin, worst-case analysis, info-gap.

1 Introduction

worst-case analysis is an extremely important concept in decision theory. It is therefore important to know whether a given decision theory is based on this concept.

As I show here, it turns out that despite its claim to be new and different from all current decision theories, *Info-Gap* is a vanilla application of *worst-case analysis*. That is, it is a simple instance of the very famous – and infamous – *Wald's Maximin Principle* (Wald, [1950])

It is therefore amazing that the concepts *worst-case analysis* and *Wald's Maximin Principle* are not discussed, let alone mentioned, in the official *Info-Gap* literature (Ben-Haim [2001, 2006]).

Unfortunately, this is just one of the things that are wrong with *Info-Gap*. Info-Gap seems to have other misconceptions about the state of the art in decision theory. More about these can be found in Sniedovich [2006].

2 Generic Info-Gap model

The generic *Info-Gap* model consists of the following ingredients:

- A decision space, $\mathbb{Q}$.
- An uncertainty space $\mathcal{U}$.
- A real valued function R on $\mathbb{Q} \times \mathcal{U}$.
- A critical reward value r_c.
- An estimate $\tilde{u}$ of a parameter $u \in \mathcal{U}$ whose true value is unknown.
- A set of nested regions of uncertainty, $\mathcal{U}(\alpha, \tilde{u}) \subseteq \mathcal{U}, \alpha \geq 0$ such that $\mathcal{U}(0, \tilde{u}) = \{\tilde{u}\}$ and $\mathcal{U}(\alpha, \tilde{u})$ is non-decreasing with α, namely $\alpha > \alpha'$ implies $\mathcal{U}(\alpha', \tilde{u}) \subseteq \mathcal{U}(\alpha, \tilde{u})$.

The *Info-Gap* recipe goes like this:

The *robustness* of decision $q \in \mathbb{Q}$ is defined as

$$\hat{\alpha}(q, r_c) := \max \left\{ \alpha \geq 0 : r_c \leq \min_{u \in \mathcal{U}(\alpha, \tilde{u})} R(q, u) \right\} \tag{1}$$

That is, it is the largest value of α such that the requirement $r_c \leq R(q, u)$ is satisfied for all $u \in \mathcal{U}(\alpha, \tilde{u})$.

The best decision is then one whose robustness is the largest. Thus, we have to solve the following optimization problem to find such a decision:

$$\hat{\alpha}(r_c) := \max_{q \in \mathbb{Q}} \ \hat{\alpha}(q, r_c) \tag{2}$$

$$= \max_{q \in \mathbb{Q}} \ \max \left\{ \alpha \geq 0 : r_c \leq \min_{u \in \mathcal{U}(\alpha, \tilde{u})} R(q, u) \right\} \tag{3}$$

In words, our mission is to find the largest value of α such that the constraint $r_c \leq R(q, u)$ is satisfied for all u in $\mathcal{U}(\alpha, \tilde{u})$ for some $q \in \mathbb{Q}$.

With no loss of generality we assume that $r_c \leq R(q, \tilde{u})$ for all $q \in \mathbb{Q}$. Any $q \in \mathbb{Q}$ that does not satisfy this requirement can be removed from $\mathbb{Q}$ at the outset.

3 Worst-case analysis and Wald's Maximin Principle

The concept *worst-case analysis* is so pervasive that it requires no explanation. Suffice to say that in classical decision theory it is deployed as the major tool for handling decision making situations under *strict uncertainty*.

The technical term used to represent this concept is *Maximin*. This term indicates that *Mother Nature* is playing "against us". That is, when we attempt to maximize our reward by controlling our decisions, *Mother Nature* always selects a state of nature that is least favorable for the decision we select. In short, we try to *maximize* the reward and *Mother Nature* is trying to *minimize* it (given our decision).

There are various ways to formulate this principle mathematically. For the purposes of this discussion it is convenient to adopt the following simple format:

$$v^* := \max_{d\in\mathbb{D}} \min_{s\in S(d)} f(d,s) \tag{4}$$

where $\mathbb{D}$ and $S(d) \subseteq \mathbb{S}, d \in \mathbb{D}$, are some sets and f is some real valued function on $\mathbb{D} \times \mathbb{S}$.

In the parlance of classical decision theory, $\mathbb{D}$ is the *decision space*, $\mathbb{S}$ is the *state space* and f is the *objective function*. The set $S(d)$ represents the set of feasible states associated with decision $d \in \mathbb{D}$.

In the classical decision theory literature this model is often called *Wald's Maximin Principle* to be distinguished from the (general) *Maximin Principle* used in *Game Theory*. The distinction is that here there is only one player – the decision maker. The second player, namely *Mother Nature*, is a modeling artifact.

A very attractive feature of this approach to severe uncertainty is that it transforms a very difficult problem of decision making under STRICT UNCERTAINTY into a very easy problem of decision making under CERTAINTY.

More details regarding the more than 50 years old *Wald's Maximin Principle* and its role in decision theory can be found in French [1988].

4 Info-Gap: a disguised worst-case analysis

To see clearly the connection between *Info-Gap* and classical *worst-case analysis*, it is convenient to define the following binary operation:

$$a \preceq b := \begin{cases} 1 & , \ a \leq b \\ 0 & , \ a > b \end{cases} \quad , \ a, b \in \mathbb{R} \tag{5}$$

where $\mathbb{R}$ denotes the real line.

We shall deploy it as indicator function of the requirement $r_c \leq R(q,u)$, observing that by definition $r_c \preceq R(q,u) = 1$ iff $r_c \leq R(q,u)$. Otherwise $r_c \preceq R(q,u) = 0$.

THEOREM. *The Info-Gap model (3) is an instance of Wald's Maximin Principle (4). That is, for any given instance of (3) there exist a collection* $W = \big(\mathbb{D}, \{S(d) \subseteq \mathbb{S} : d \in \mathbb{D}\}, f\big)$ *such that (3) is equivalent to (4).*

Proof. By definition

$$\hat{\alpha}(r_c) = \max_{q\in\mathbb{Q}} \max \left\{\alpha : r_c \leq \min_{u\in\mathcal{U}(\alpha,\tilde{u})} R(q,u)\right\} \tag{6}$$

$$= \max_{q\in\mathbb{Q},\alpha\geq 0} \left\{\alpha : r_c \leq \min_{u\in\mathcal{U}(\alpha,\tilde{u})} R(q,u)\right\} \tag{7}$$

Hence,

$$\hat{\alpha}(r_c) = \max_{q\in\mathbb{Q},\alpha\geq 0} \alpha \left(r_c \preceq \min_{u\in\mathcal{U}(\alpha,\tilde{u})} R(q,u)\right) \tag{8}$$

$$= \max_{q\in\mathbb{Q},\alpha\geq 0} \min_{u\in\mathcal{U}(\alpha,\tilde{u})} \alpha\left(r_c \preceq R(q,u)\right) \tag{9}$$

Thus, the correspondence between the two models is as shown in Table 1, where $\mathbb{R}_+ := \{r \in \mathbb{R} : r \geq 0\}$. $\mathfrak{QED}$

Wald	*Info-Gap*
d	(q,α)
s	u
$f(d,s)$	$\alpha\left(r_c \preceq R(q,u)\right)$
$\mathbb{D}$	$\mathbb{Q}\times\mathbb{R}_+$
$S(d)$	$\mathcal{U}(q,\tilde{u})$
$\mathbb{S}$	$\bigcup_{\alpha\geq 0}\mathcal{U}(\alpha,\tilde{u})$

Table 1: Correspondence between Info-Gap and Wald's Maximin Principle

5 Discussion

The "discovery" that *Info-Gap* is a disguised worst-case analysis is a bit surprising because *Info-Gap* itself claims that in the framework of *Info-Gap* there is no worst case:

> The info-gap model is unbounded in the sense that there is no largest set and there is no worst case.
>
> Carmel and Ben-Haim [2005, p. 635]

> It is important to emphasize that the robustness $h(R,c)$ is *not* a minimax algorithm. In minimax robustness analysis, one *minimizes* the *maximum* adversity. This is not what info-gap robustness does. There is no maximal adversity in an info-gap model of uncertainty: the worst case at any horizon of uncertainty h is less damaging than some realization at a greater

horizon of uncertainty. Since the horizon of of uncertainty is unbounded, there is no worst case and the info-gap analysis cannot and does not purport to ameliorate a worst case .

Ben-Haim [2005, p. 392]

Perhaps this explain the rather puzzling fact that the concept "worst-case analysis" and the famous *Wald's Maximin Principle* are not discussed at all in the official *Info-Gap* literature (Ben-Haim [2001, 2006]).

Observe that in accordance with the *Maximin* model, *Mother Nature* selects the worst element of the region $\mathcal{U}(\alpha, \tilde{u})$, hence there is definitely a worst-case analysis here.

The fact that no upper bound on α is specified (and therefore the total region of uncertainty $\mathcal{U}$ can be unbounded) should not be confused with the fact that within each region $\mathcal{U}(\alpha, \tilde{u})$ the objective function may possess a worst case. Indeed, this is the usual case rather than the exception.

More fundamentally, whether a worst case exists on the unbounded total region of uncertainty $\mathcal{U}$ depends on the objective function used. This function can be bounded on $\mathcal{U}$ even tough $\mathcal{U}$ is unbounded[1].

As we demonstrated above, this behavior can be easily incorporated in *Wald's Maximin Principle* to allow it to cope with *Info-Gap* as an instance, rather than as an exception.

A more detailed analysis of the misconceptions *Info-Gap* apparently has concerning *worst-case analysis* and consequently *Wald's Maximin Principle* can be found in Sniedovich [2006].

6 Conclusion

Pronouncements such as

> Info-gap decision theory is radically different from all current theories of decision under uncertainty. The difference originates in the modelling of uncertainty as an information gap rather than as a probability.
>
> Ben-Haim [2006, p.xii]

and

> In this book we concentrate on the fairly new concept of information-gap uncertainty, whose differences from more classical approaches to uncertainty are real and deep. Despite the power of classical decision theories, in many areas such as engineering, economics, management, medicine and public policy, a need has arisen for a different format for decisions based on severely uncertain evidence.
>
> Ben-Haim [2006, p. 11]

[1] *Info-gap* seems to suggest that just because the domain of a function is unbounded, the function itself is unbounded on this domain. This is of course not so, eg. $\sin(x)$ is bounded on the (unbounded) real line.

reveal serious misconceptions that *Info-Gap* has about classical decision theory.

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